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Fusion rules of the $\mathcal{W}_{p,q}$ triplet models

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Abstract

In this paper we determine the fusion rules of the logarithmic $\mathcal{W}_{p,q}$ triplet theory and construct the Grothendieck group with subgroups for which consistent product structures can be defined. The fusion rules are then used to determine projective covers. This allows us also to write down a candidate for a modular invariant partition function. Our results demonstrate that recent work on the $\mathcal{W}_{2,3}$ model generalizes naturally to arbitrary (p, q) .

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1. Introduction

Logarithmic conformal field theories appear in the description of critical points in many interesting physical systems. Some examples are polymers, spin chains, percolation and sand-pile models, see for example [1–9] for some recent papers. A lot of effort has been invested recently to try and understand these theories in a general context. For example the logarithmic conformal field theories from the $(1, p)$ -series have been studied in quite some detail and their structure is now largely understood [10–16]. However, the more general (p, q) -series for p, q coprime and $p, q \geq 2$ are not as well understood yet, though there has been some progress recently [17–20]. These theories are referred to as the $\mathcal{W}_{p,q}$ triplet models, and they can be naturally associated with the minimal models for $p, q \geq 2$.

The goal of this paper is to generalize the results of [17], where the $\mathcal{W}_{2,3}$ model was studied, to general (p, q) . In particular we obtain the fusion rules of the $\mathcal{W}_{p,q}$ triplet models. This is obviously a prerequisite for any detailed analysis of this theory. We also study the Grothendieck group that plays a vital role in the boundary description of conformal field theories. One novel feature of the $\mathcal{W}_{p,q}$ models is that the vacuum representation is reducible but indecomposable and we believe that it is responsible for the fact that the structure of the $\mathcal{W}_{p,q}$ models, though closely related to that of the minimal models and the $\mathcal{W}_{1,p}$ models, is a lot more complicated than either.

We determine the fusion rules by first generalizing the representations appearing in [17] for arbitrary p, q in section 2. For a subset of these representations the fusion rules have

already been determined in [18, 19] and we will propose a way to extend these rules in an associative manner to all the other representations in section 3 (with consistency checks in appendix E for certain explicit values of (p, q)). In section 4 we determine the subgroup of the Grothendieck group for which a consistent product structure induced by the fusion rules can be defined. In section 5 we address the problem of determining the projective representations among our representations and suggest a candidate for a modular invariant bulk spectrum.

2. Representations and their structure

We begin with a quick review of minimal models and their generalization to logarithmic theories; for details on our notation please consult appendix A. The Virasoro (non-logarithmic) minimal models and $\mathcal{W}_{p,q}$ -models are labelled by coprime positive integers (p, q) and have central charge

$$c_{p,q} = 1 - 6 \frac{(p - q)^2}{pq}. \tag{2.1}$$

The irreducible representations have weights

$$h_{m,n} = \frac{(pn - qm)^2 - (p - q)^2}{4pq}, \tag{2.2}$$

where m and n are positive integers, and $h_{m,n} = h_{p-m,q-n}$.

The non-logarithmic minimal models are representations of the vertex operator algebra (VOA) also known as the vacuum representation $\mathcal{V}(h_{1,1} = 0)$. This is the irreducible highest weight representation of the Virasoro algebra based on the highest weight state Ω with weight $h = 0$. The corresponding Verma module has two nullvectors \mathcal{N}_1 and $\mathcal{N}_{(p-1)(q-1)}$ at levels 1 and $(p - 1)(q - 1)$ respectively. Setting both nullvectors to zero one obtains the irreducible vacuum representation based on Ω . The highest weight representations $\mathcal{V}(h_{a,b})$ of the VOA with weight $h_{a,b}$ are the representations of the Virasoro algebra for which the modes of the vertex operators $V(\mathcal{N}_1, z)$ and $V(\mathcal{N}_{(p-1)(q-1)}, z)$ act trivially.

The logarithmic theories of interest in this paper are constructed by only quotienting out the nullvector at level 1 in the Verma module corresponding to the VOA but not the nullvector at level $(p - 1)(q - 1)$. This prevents the the VOA from being irreducible, but it is still indecomposable. The corresponding theory is not rational, however, since the fusion of irreducible representations of this VOA no longer closes on a finite set. The repeated fusion of irreducibles produces an infinite series of irreducible representations with weights of the form (2.2) as well as reducible but indecomposable combinations of these irreducible representations. To restore rationality the chiral algebra is enlarged by three fields of conformal weight $(2p - 1)(2q - 1)$. We denote the resulting VOA by $\mathcal{W}(p, q)$, its irreducible highest weight representations of weight h are denoted by $\mathcal{W}(h)$. The fusion of irreducible representations of $\mathcal{W}(p, q)$ closes on a finite set, but apart from an irreducible representation for every weight of the form (A.1), this set also includes reducible but indecomposable combinations of these representations.

2.1. Representation content

The $\mathcal{W}_{p,q}$ -models close under the conjectured fusion rules of a grand total of $4pq + 13 \frac{(p-1)(q-1)}{2} - 2$ representations. This is the smallest such set of representations, containing the vertex operator algebra (VOA) and the irreducible representations. For convenience, we group these representations into two lists B and N. The labelling of the weights in the following lists is explained in (A.1).

Representations of type B

- $2(p + q - 1)$ irreducible representations:

$$\mathcal{W}(h_{(r,q,\pm)}) \quad \text{and} \quad \mathcal{W}(h_{(p,s,\pm)}). \tag{2.3}$$

It was shown in [20] that these irreducible representations can be interpreted as infinite sums of Virasoro representations.

- $4pq - 2(p + q)$ rank 2 representations which are reducible but indecomposable and whose L_0 action is not diagonalizable but rather contains 2×2 Jordan blocks:

$$\begin{aligned} \mathcal{R}^{(2)}(h_{(a,b,+)}; h_{(p-a,b,-)}) & \quad \mathcal{R}^{(2)}(h_{(a,b,+)}; h_{(a,q-b,-)}) \\ \mathcal{R}^{(2)}(h_{(a,q-b,-)}; h_{(a,b,+)}) & \quad \mathcal{R}^{(2)}(h_{(p-a,b,-)}; h_{(a,b,+)}) \\ \mathcal{R}^{(2)}(h_{(a,q,+)}) & \quad \mathcal{R}^{(2)}(h_{(p-a,b,-)}) \\ \mathcal{R}^{(2)}(h_{(p,b,+)}) & \quad \mathcal{R}^{(2)}(h_{(p,q-b,-)}). \end{aligned} \tag{2.4}$$

The first entry in $\mathcal{R}^{(2)}(h_1; h_2)$ or $\mathcal{R}^{(2)}(h)$ is the weight of the cyclic vector that generates the entire representation. For weights of type $h_{(a,b,\pm)}$ this does not uniquely determine the rank 2 representation and an extra weight $h_{(a',b',\mp)}$ is required to specify the representation in question.

- $2(p - 1)(q - 1)$ rank 3 representations which are reducible but indecomposable and whose L_0 action is not diagonalizable but rather contains 3×3 Jordan blocks:

$$\mathcal{R}^{(3)}(h_{(a,b,\pm)}). \tag{2.5}$$

Here the argument h of $\mathcal{R}^{(3)}(h)$ is the weight of the generating cyclic state.

Note that the rank 2 and 3 representations are obtained by repeated products of the irreducible representations.

Representations of type N

- $\frac{1}{2}(p - 1)(q - 1)$ irreducible highest weight representations coming from the non-logarithmic minimal model:

$$\mathcal{W}(h_{(a,b,0)}) = \mathcal{W}(h_{(p-a,q-b,0)}). \tag{2.6}$$

It was shown in [20] that these irreducible representations are just the irreducible Virasoro representations of the same weight.

- $(p - 1)(q - 1)$ rank 1 highest weight representations

$$\mathcal{W}_{a,b}, \tag{2.7}$$

which are reducible but indecomposable and whose L_0 action is diagonalizable. These representations were also introduced in [19].

- $(p - 1)(q - 1)$ conjugates of the rank 1 representations $\mathcal{W}_{a,b}$ which we shall denote by

$$\mathcal{W}_{a,b}^*. \tag{2.8}$$

- $2(p - 1)(q - 1)$ irreducible highest weight representations with weights that are descendants of those appearing in the non-logarithmic minimal models:

$$\mathcal{W}(h_{(a,b,\pm)}). \tag{2.9}$$

It was shown in [20] that these irreducible representations can be interpreted as infinite sums of Virasoro representations.

By the same arguments as in [17] we believe that the representations of type B are those that define a consistent boundary theory and therefore also show up in lattice considerations such as [18], where fusion rules for representations of this type are presented. For these representations the notion of duals and contragredients is the same (in fact these representations are self-contragredient and therefore also self-dual); see [17] section 3 for further details on duals and contragredients. It is also reassuring to note that this generalization matches the representations appearing in [18] exactly. By contrast the representations of type N do not define a consistent boundary theory, and, with the exception of the representations of type $\mathcal{W}_{a,b}$, the fusion rules for these representations are not yet known. The goal of this paper is to extend the fusion rules to include all representations of type N.

The representations of type $\mathcal{W}(h_{(a,b,0)})$ are a somewhat special class of representations in N and have to be considered separately in a number of cases in our analysis. We will therefore restrict ourselves to N_\times , the set of all representations of types $\mathcal{W}_{a,b}$, $\mathcal{W}_{a,b}^*$ or $\mathcal{W}(h_{(a,b,\pm)})$, whenever we need to temporarily exclude the representations of type $\mathcal{W}(h_{(a,b,0)})$ from our considerations. In order to be able to extend the fusion rules to N_\times , we need to understand the detailed structure of representations of types $\mathcal{W}_{a,b}$ and $\mathcal{W}_{a,b}^*$ a little better.

- The representations of type $\mathcal{W}_{a,b}$ correspond to weight $h_{(a,b,0)}$ Verma modules where only the nullvector at level ab is quotiented out, but not the nullvector at level $(p-a)(q-b)$. They are characterized by the short exact sequences

$$0 \longrightarrow \mathcal{W}(h_{(a,b,+)}) \longrightarrow \mathcal{W}_{a,b} \longrightarrow \mathcal{W}(h_{(a,b,0)}) \longrightarrow 0, \tag{2.10}$$

i.e. $\mathcal{W}(h_{(a,b,+)})$ is a subrepresentation of $\mathcal{W}_{a,b}$, which implies

$$\mathcal{W}(h_{(a,b,0)}) = \mathcal{W}_{a,b}/\mathcal{W}(h_{(a,b,+)}). \tag{2.11}$$

In particular the fusion of $\mathcal{W}_{a,b}$ is the same as that of $\mathcal{W}(h_{(a,b,0)})$ with all representations whose fusion with $\mathcal{W}(h_{(a,b,+)})$ vanishes. This class of representations includes the VOA $\mathcal{W}(p, q) \equiv \mathcal{W}_{1,1}$.

- The representations of type $\mathcal{W}_{a,b}^*$ are generated by cyclic vectors of weight $h_{(a,b,+)}$. These cyclic vectors are not highest weight however. We conjecture in analogy to [17] that the positive modes of the chiral algebra map the cyclic vector to a vector that generates $\mathcal{W}(h_{(a,b,0)})$ as a subrepresentation and that representations of type $\mathcal{W}_{a,b}^*$ are characterized by the short exact sequences

$$0 \longrightarrow \mathcal{W}(h_{(a,b,0)}) \longrightarrow \mathcal{W}_{a,b}^* \longrightarrow \mathcal{W}(h_{(a,b,+)}) \longrightarrow 0. \tag{2.12}$$

This implies

$$\mathcal{W}(h_{(a,b,+)}) = \mathcal{W}_{a,b}^*/\mathcal{W}(h_{(a,b,0)}). \tag{2.13}$$

In particular the fusion of $\mathcal{W}_{a,b}^*$ is the same as that of $\mathcal{W}(h_{(a,b,+)})$ with all representations whose fusion with $\mathcal{W}(h_{(a,b,0)})$ vanishes.

3. Fusion rules

As mentioned in the introduction the fusion rules for representations of type B as well as representations of type $\mathcal{W}_{a,b}$ have already been determined in [18, 19]. For the representations of type $\mathcal{W}(h_{(a,b,0)})$ the fusion rules are given by the minimal model fusion rules which have already been known for quite some time [21].

We will now extend the the fusion rules to include all representations of type N, by first considering products of representations of type N_\times with representations of types B or N_\times , before considering representations of type $\mathcal{W}(h_{(a,b,0)})$. The general strategy is to rewrite

all representations in N as the the fusion product of a representations of type $\mathcal{W}_{a,b}$ and $\mathcal{W}(h_{(1,1,0)})$, $\mathcal{W}(h_{(1,1,+)})$, $\mathcal{W}(h_{(1,1,-)})$ or $\mathcal{W}_{1,1}^*$. Using commutativity and associativity of the fusion product together with conjectured fusion rules for $\mathcal{W}(h_{(1,1,0)})$, $\mathcal{W}(h_{(1,1,+)})$, $\mathcal{W}(h_{(1,1,-)})$ and $\mathcal{W}_{1,1}^*$, we can then define fusion rules for all of N .

3.1. Products involving representations of type N_\times

3.1.1. The conjectured fusion rules for $\mathcal{W}(h_{(1,1,+)})$, $\mathcal{W}(h_{(1,1,-)})$ and $\mathcal{W}_{1,1}^*$. In order to extend the fusion rules to representations of type N_\times , we first need to understand the fusion rules of $\mathcal{W}(h_{(1,1,+)})$, $\mathcal{W}(h_{(1,1,-)})$ and $\mathcal{W}_{1,1}^*$. In analogy to [17] and inspired by [20], we conjecture that $\mathcal{W}(h_{(1,1,+)})$, $\mathcal{W}(h_{(1,1,-)})$ and $\mathcal{W}_{1,1}^*$ obey the following fusion rules:

$$\begin{aligned} \mathcal{W}(h_{(1,1,+)}) \otimes \mathcal{W}(h_{(1,1,+)}) &= \mathcal{W}_{1,1}^* & \mathcal{W}(h_{(1,1,-)}) \otimes \mathcal{W}(h_{(1,1,-)}) &= \mathcal{W}_{1,1}^* \\ \mathcal{W}(h_{(1,1,+)}) \otimes \mathcal{W}(h_{(1,1,-)}) &= \mathcal{W}(h_{(1,1,-)}) \end{aligned} \tag{3.1}$$

In appendix E we check $\mathcal{W}(h_{(1,1,+)}) \otimes \mathcal{W}(h_{(1,1,+)})$ for a number of cases using the NGK-algorithm introduced in [22, 23]. From (3.1) we can derive the remaining three products using associativity and the quotient (2.13):

$$\mathcal{W}(h_{(1,1,+)}) \otimes \mathcal{W}_{1,1}^* \stackrel{(2.13)}{=} \mathcal{W}(h_{(1,1,+)}) \otimes \mathcal{W}(h_{(1,1,+)}) = \mathcal{W}_{1,1}^* \tag{3.2}$$

$$\begin{aligned} \mathcal{W}_{1,1}^* \otimes \mathcal{W}_{1,1}^* &\stackrel{(3.1)}{=} \mathcal{W}(h_{(1,1,+)}) \otimes \mathcal{W}(h_{(1,1,+)}) \otimes \mathcal{W}_{1,1}^* \\ &\stackrel{(3.2)}{=} \mathcal{W}(h_{(1,1,+)}) \otimes \mathcal{W}_{1,1}^* \stackrel{(3.2)}{=} \mathcal{W}_{1,1}^* \end{aligned} \tag{3.3}$$

$$\begin{aligned} \mathcal{W}_{1,1}^* \otimes \mathcal{W}(h_{(1,1,-)}) &\stackrel{(3.1)}{=} \mathcal{W}(h_{(1,1,+)}) \otimes \mathcal{W}(h_{(1,1,+)}) \otimes \mathcal{W}(h_{(1,1,-)}) \\ &\stackrel{(3.1)}{=} \mathcal{W}(h_{(1,1,+)}) \otimes \mathcal{W}(h_{(1,1,-)}) = \mathcal{W}(h_{(1,1,-)}) \end{aligned} \tag{3.4}$$

The use of the quotient (2.13) is justified, because $\mathcal{W}(h_{(1,1,0)}) \otimes \mathcal{W}(h_{(1,1,+)}) = 0$ as we will see in (3.17). Again in analogy to [17], we conjecture that for representations of type $\mathcal{W}_{a,b}$ the fusion with the representations $\mathcal{W}(h_{(1,1,+)})$, $\mathcal{W}(h_{(1,1,-)})$ and $\mathcal{W}_{1,1}^*$ is given by

$$\begin{aligned} \mathcal{W}_{1,1}^* \otimes \mathcal{W}_{a,b} &= \mathcal{W}_{a,b}^*, & \mathcal{W}(h_{(1,1,+)}) \otimes \mathcal{W}_{a,b} &= \mathcal{W}(h_{(a,b,+)}), \\ \mathcal{W}(h_{(1,1,-)}) \otimes \mathcal{W}_{a,b} &= \mathcal{W}(h_{(a,b,-)}). \end{aligned} \tag{3.5}$$

This is enough information to determine the product of the representations of type N_\times using the fusion rules for two representations of type $\mathcal{W}_{a,b}$ listed in appendix C.

To determine the product of a representation of type N_\times and a representation of type B, it is sufficient to conjecture (also in analogy to [17]) the action of $\mathcal{W}(h_{(1,1,+)})$ and $\mathcal{W}(h_{(1,1,-)})$ on the irreducible representations

$$\begin{aligned} \mathcal{W}(h_{(1,1,+)}) \otimes \mathcal{W}(h_{(r,q,\pm)}) &= \mathcal{W}(h_{(r,q,\pm)}), & \mathcal{W}(h_{(1,1,+)}) \otimes \mathcal{W}(h_{(p,s,\pm)}) &= \mathcal{W}(h_{(p,s,\pm)}), \\ \mathcal{W}(h_{(1,1,-)}) \otimes \mathcal{W}(h_{(r,q,+)}) &= \mathcal{W}(h_{(r,q,-)}), & \mathcal{W}(h_{(1,1,-)}) \otimes \mathcal{W}(h_{(p,s,+)}) &= \mathcal{W}(h_{(p,s,-)}), \end{aligned} \tag{3.6}$$

since the rank 2 and rank 3 representations are products of the irreducible representations. On representations of type B the action of $\mathcal{W}_{1,1}^*$ is the same as that of $\mathcal{W}(h_{(1,1,+)})$, because of (3.1) and associativity.

In summary the $3pq$ conjectured products (3.1), (3.5) and (3.6) are sufficient to extend the fusion rules to N_\times by associativity and commutativity.

3.1.2. *Closed fusion formula for N_\times .* In an attempt to improve readability, we now introduce some more notation that will help us apply (3.1), (3.5) and (3.6) to arbitrary products. For every label (a, b) we associate four representations of type N_\times :

$$\begin{array}{ccc}
 & \mathcal{W}_{a,b} & \\
 \nearrow \mathfrak{N}_+ & \mathfrak{B} & \nwarrow \mathfrak{N}_- \\
 \mathcal{W}(h_{(a,b,+)}) & \mathfrak{B} & \mathcal{W}(h_{(a,b,-)}) \\
 \searrow \mathfrak{N}_+ & \mathfrak{N}_* & \nearrow \mathfrak{N}_- \\
 & \mathcal{W}_{a,b}^* &
 \end{array}
 \tag{3.7}$$

The four labels $\mathfrak{N}_{\pm,*}$ and \mathfrak{B} of the arrows are maps from the set of representations to itself, that are linear with respect to direct sums, i.e. for a sum of representations they are evaluated for each representation separately.

- (1) \mathfrak{B} maps representations of type N_\times to their corresponding representation in (3.7) of type $\mathcal{W}_{a,b}$:

$$\mathfrak{B}(\mathcal{W}_{a,b}) = \mathfrak{B}(\mathcal{W}_{a,b}^*) = \mathfrak{B}(\mathcal{W}(h_{(a,b,+)})) = \mathfrak{B}(\mathcal{W}(h_{(a,b,-)})) = \mathcal{W}_{a,b},
 \tag{3.8}$$

and acts as identity for representations of type B. The image of \mathfrak{B} is therefore the representations of type B and representations of type $\mathcal{W}_{a,b}$. For these representations the fusion rules have already been determined in [18, 19].

- (2) The map \mathfrak{N}_* corresponds to the action of $\mathcal{W}_{1,1}^*$ in [17], generalized for arbitrary (p, q) . It maps the representations in (3.7) to

$$\begin{aligned}
 \mathfrak{N}_*(\mathcal{W}_{a,b}) &= \mathfrak{N}_*(\mathcal{W}_{a,b}^*) = \mathfrak{N}_*(\mathcal{W}(h_{(a,b,+)})) = \mathcal{W}_{a,b}^* \\
 \mathfrak{N}_*(\mathcal{W}(h_{(a,b,-)})) &= \mathcal{W}(h_{(a,b,-)}),
 \end{aligned}
 \tag{3.9}$$

and acts as the identity on representations of type B. This corresponds precisely to the fusion products in (3.2)–(3.4).

- (3) The map \mathfrak{N}_+ corresponds to the action of $\mathcal{W}(h_{(1,1,+)})$ in [17], generalized for arbitrary (p, q) . It maps the representations in (3.7) to

$$\begin{aligned}
 \mathfrak{N}_+(\mathcal{W}_{a,b}) &= \mathcal{W}(h_{(a,b,+)}) \\
 \mathfrak{N}_+(\mathcal{W}_{a,b}^*) &= \mathfrak{N}_+(\mathcal{W}(h_{(a,b,+)})) = \mathcal{W}_{a,b}^* \\
 \mathfrak{N}_+(\mathcal{W}(h_{(a,b,-)})) &= \mathcal{W}(h_{(a,b,-)})
 \end{aligned}
 \tag{3.10}$$

and acts as the identity on representations of type B. This corresponds precisely to the fusion products in (3.1).

- (4) The map \mathfrak{N}_- corresponds to the action of $\mathcal{W}(h_{(1,1,-)})$ in [17], generalized for arbitrary (p, q) . It maps the representations in (3.7) to

$$\begin{aligned}
 \mathfrak{N}_-(\mathcal{W}_{a,b}) &= \mathfrak{N}_-(\mathcal{W}_{a,b}^*) = \mathfrak{N}_-(\mathcal{W}(h_{(a,b,+)})) = \mathcal{W}(h_{(a,b,-)}) \\
 \mathfrak{N}_-(\mathcal{W}(h_{(a,b,-)})) &= \mathcal{W}_{a,b}^*,
 \end{aligned}
 \tag{3.11}$$

and on representations of type B it exchanges the weights $h_{(r,s,\pm)}$ by $h_{(r,s,\mp)}$:

$$\begin{aligned}
 \mathfrak{N}_-(\mathcal{W}(h_{(r,q,\pm)})) &= \mathcal{W}(h_{(r,q,\mp)}), & \mathfrak{N}_-(\mathcal{W}(h_{(p,s,\pm)})) &= \mathcal{W}(h_{(p,s,\mp)}), \\
 \mathfrak{N}_-(\mathcal{R}^{(2)}(h_{(r,q,\pm)})) &= \mathcal{R}^{(2)}(h_{(r,q,\mp)}), & \mathfrak{N}_-(\mathcal{R}^{(2)}(h_{(p,s,\pm)})) &= \mathcal{R}^{(2)}(h_{(p,s,\mp)}), \\
 \mathfrak{N}_-(\mathcal{R}^{(2)}(h_{(a,b,\pm)}; h_{(a',b',\mp)})) &= \mathcal{R}^{(2)}(h_{(a,b,\mp)}; h_{(a',b',\pm)}), \\
 \mathfrak{N}_-(\mathcal{R}^{(3)}(h_{(a,b,\pm)})) &= \mathcal{R}^{(3)}(h_{(a,b,\mp)}).
 \end{aligned}$$

This corresponds precisely to the fusion products in (3.1).

By straightforward computation, we see that $\mathfrak{N}_{\pm,*}$ and \mathfrak{B} satisfy the following composition rules:

$$\begin{aligned} \mathfrak{N}_* \circ \mathfrak{N}_* &= \mathfrak{N}_*, & \mathfrak{N}_+ \circ \mathfrak{N}_+ &= \mathfrak{N}_*, & \mathfrak{N}_- \circ \mathfrak{N}_- &= \mathfrak{N}_*, & (3.12) \\ \mathfrak{N}_* \circ \mathfrak{N}_+ &= \mathfrak{N}_+ \circ \mathfrak{N}_* = \mathfrak{N}_*, & \mathfrak{N}_* \circ \mathfrak{N}_- &= \mathfrak{N}_- \circ \mathfrak{N}_* = \mathfrak{N}_-, \\ \mathfrak{N}_- \circ \mathfrak{N}_+ &= \mathfrak{N}_+ \circ \mathfrak{N}_- = \mathfrak{N}_-, \\ \mathfrak{B} \circ \mathfrak{N}_{\pm,*} &= \mathfrak{B}, & \mathfrak{N}_{\pm,*} \circ \mathfrak{B} &= \mathfrak{N}_{\pm,*}. \end{aligned}$$

As a final piece of notation, when we write \mathfrak{N}_A for some representation A , we mean

$$\mathfrak{N}_A = \begin{cases} \mathfrak{N}_* & A = \mathcal{W}_{a,b}^* \\ \mathfrak{N}_+ & A = \mathcal{W}(h_{(a,b,+)} \\ \mathfrak{N}_- & A = \mathcal{W}(h_{(a,b,-)} \\ \text{id} & \text{else.} \end{cases} \quad (3.13)$$

This notation allows us to write

$$A = \mathfrak{N}_A \circ \mathfrak{B}(A). \quad (3.14)$$

This is just another way of writing (3.5) in terms of \mathfrak{N} and \mathfrak{B} .

Armed with all the information and notation of this section, we can then see that the product of two indecomposable representations A and B (remember that we are not yet considering representations of type $\mathcal{W}(h_{(a,b,0)})$) is given by

$$A \otimes B = \mathfrak{N}_A \circ \mathfrak{N}_B(\mathfrak{B}(A) \otimes \mathfrak{B}(B)). \quad (3.15)$$

In effect, this is simply rewriting the product of A and B in term products involving $\mathcal{W}(h_{(1,1,+)}), \mathcal{W}(h_{(1,1,-)})$ and $\mathcal{W}_{1,1}^*$ as well as the known product $\mathfrak{B}(A) \otimes \mathfrak{B}(B)$. The product $\mathfrak{B}(A) \otimes \mathfrak{B}(B)$ is evaluated using the fusion rules in [18, 19] and $\mathfrak{N}_A \circ \mathfrak{N}_B$ is then applied to the result. As an example, we will compute $\mathcal{W}(h_{(1,2,+)} \otimes \mathcal{W}(h_{(1,2,-)})$ for $q \geq 3$. Using (3.15) we find,

$$\begin{aligned} \mathcal{W}(h_{(1,2,+)} \otimes \mathcal{W}(h_{(1,2,-)})) &= \mathfrak{N}_{\mathcal{W}(h_{(1,2,+)})} \circ \mathfrak{N}_{\mathcal{W}(h_{(1,2,-)})}(\mathcal{W}_{1,2} \otimes \mathcal{W}_{1,2}) = \mathfrak{N}_+ \circ \mathfrak{N}_-(\mathcal{W}_{1,2} \otimes \mathcal{W}_{1,2}) \\ &= \mathfrak{N}_+ \circ \mathfrak{N}_-(\mathcal{W}_{1,1} \oplus \mathcal{W}_{1,3}) = \mathfrak{N}_-(\mathcal{W}_{1,1} \oplus \mathcal{W}_{1,3}) \\ &= \mathfrak{N}_-(\mathcal{W}_{1,1}) \oplus \mathfrak{N}_-(\mathcal{W}_{1,3}) = \mathcal{W}(h_{(1,1,-)}) \oplus \mathcal{W}(h_{(1,3,-)}), \end{aligned}$$

where $\mathcal{W}_{1,2} \otimes \mathcal{W}_{1,2}$ was evaluated using (C.1). It is easy to check that (3.15) agrees with (3.1)–(3.6). We will show in section 3.3 that it leads to associative fusion rules.

3.2. Products involving representations of type $\mathcal{W}(h_{(a,b,0)})$

The final step towards extending the fusion rules to all representations of type N is to consider products involving representations of type $\mathcal{W}(h_{(a,b,0)})$. As a first step we consider products of the form $\mathcal{W}(h_{(1,1,0)}) \otimes \mathcal{W}(h_{(r,s,\pm)})$. As mentioned before, the representations of type $\mathcal{W}(h_{(a,b,0)})$ come from the non-logarithmic minimal model and satisfy the minimal model fusion rules among themselves:

$$\mathcal{W}(h_{(a,b,0)}) \otimes \mathcal{W}(h_{(a',b',0)}) = \sum_{\substack{k=1+|a-a'| \\ k+a+a'=1 \pmod 2}}^{\min\{a+a'-1, 2p-1-a-a'\}} \sum_{\substack{l=1+|b-b'| \\ l+b+b'=1 \pmod 2}}^{\min\{b+b'-1, 2q-1-b-b'\}} \mathcal{W}(h_{(k,l,0)}). \quad (3.16)$$

Since $\mathcal{W}(h_{(1,1,0)})$ is the VOA of the non-logarithmic minimal model, its fusion acts as the identity on representations in the non-logarithmic minimal model and the product with any other irreducible representations vanishes. Therefore, we have

$$\mathcal{W}(h_{(1,1,0)}) \otimes \mathcal{W}(h_{(r,s,\pm)}) = 0. \quad (3.17)$$

This also ties in with what one would expect from the NGK-algorithm in [22, 23] on the level of the Virasoro algebra¹.

Using associativity and the fact that $\mathcal{W}(h_{(1,1,0)})$ acts as the identity on representations of type $\mathcal{W}(h_{(a,b,0)})$, we can compute a more general version of (3.17):

$$\mathcal{W}(h_{(a,b,0)}) \otimes \mathcal{W}(h_{(r,s,\pm)}) = \mathcal{W}(h_{(a,b,0)}) \otimes \mathcal{W}(h_{(1,1,0)}) \otimes \mathcal{W}(h_{(r,s,\pm)}) = 0. \quad (3.18)$$

Since the rank 2 and 3 representations are just products of irreducible representations of type B, fusion of minimal model representations $\mathcal{W}(h_{(a,b,0)})$ with these vanishes as well. Therefore, associativity guarantees that

$$\mathcal{W}(h_{(a,b,0)}) \otimes \text{Representation of type B} = 0. \quad (3.19)$$

This specifies the fusion rules of representations of type $\mathcal{W}(h_{(a,b,0)})$ with all representations in B, as well as the irreducible representations in N. All that remains is to describe products of $\mathcal{W}(h_{(a,b,0)})$ with $\mathcal{W}_{a,b}$ or $\mathcal{W}_{a,b}^*$. Using associativity (3.2) and (3.5), we see that $\mathcal{W}(h_{(a,b,0)}) \otimes \mathcal{W}_{a',b'}^*$ can be written as

$$\begin{aligned} \mathcal{W}(h_{(a,b,0)}) \otimes \mathcal{W}_{a',b'}^* &= \mathcal{W}(h_{(a,b,0)}) \otimes \mathcal{W}_{1,1}^* \otimes \mathcal{W}_{a',b'} \\ &= \mathcal{W}(h_{(a,b,0)}) \otimes \mathcal{W}(h_{(1,1,+)}) \otimes \mathcal{W}(h_{(1,1,+)}) \otimes \mathcal{W}_{a',b'} = 0 \end{aligned} \quad (3.20)$$

and therefore the fusion of all $\mathcal{W}_{a',b'}^*$ with all $\mathcal{W}(h_{(a,b,0)})$ vanishes. Products of $\mathcal{W}(h_{(a,b,0)})$ with representations of type $\mathcal{W}_{a,b}$, can be computed using the quotient (2.11)

$$\mathcal{W}(h_{(a,b,0)}) \otimes \mathcal{W}_{a',b'} = \mathcal{W}(h_{(a,b,0)}) \otimes \mathcal{W}(h_{(a',b',0)}). \quad (3.21)$$

In summary, we therefore have that the fusion rules of $\mathcal{W}(h_{(a,b,0)})$ satisfy the following.

- (1) All products of $\mathcal{W}(h_{(a,b,0)})$ with representations not of type $\mathcal{W}_{a,b}$ or $\mathcal{W}(h_{(a,b,0)})$ vanish.
- (2) Products of $\mathcal{W}(h_{(a,b,0)})$ with representations of type $\mathcal{W}_{a,b}$ or $\mathcal{W}(h_{(a,b,0)})$ are given by the non-logarithmic minimal model fusion rules.

3.3. Associativity

Now that we have extended the fusion rules to all representations of type N, we still need to prove that they are associative. We do this by considering three cases. First we consider products of $\mathcal{W}(h_{(a,b,0)})$ with representations of type $\mathcal{W}_{a,b}$ or $\mathcal{W}(h_{(a,b,0)})$, second we consider products of $\mathcal{W}(h_{(a,b,0)})$ with anything else and third, products not involving representations of type $\mathcal{W}(h_{(a,b,0)})$.

- (1) As we discovered in the previous section, products involving a representation of type $\mathcal{W}(h_{(a,b,0)})$ together with representations of type $\mathcal{W}_{a,b}$ or $\mathcal{W}(h_{(a,b,0)})$ are given by the non-logarithmic minimal model fusion rules. These are known to be associative.
- (2) As one can see from formula (3.15) and the definitions of \mathfrak{N}_+ , \mathfrak{N}_- and \mathfrak{N}_* , products involving representations of type B, $\mathcal{W}(h_{(a,b,\pm)})$ or $\mathcal{W}_{a,b}^*$ will never contain a representation of type $\mathcal{W}_{a,b}$ or $\mathcal{W}(h_{(a,b,0)})$ as a summand in their result. Therefore, all products involving representations of type $\mathcal{W}(h_{(a,b,0)})$ and representations not of type $\mathcal{W}_{a,b}$ or $\mathcal{W}(h_{(a,b,0)})$ vanish, regardless of the order in which the product is computed; hence, this case is also associative.

¹ One sees that the quotient space of vectors not lying in the image of words with negative L_0 -grading is at most one-dimensional because $\mathcal{W}(h_{(1,1,0)})$ has the nullvector $\mathcal{N}_1 = L_{-1}\Omega$ at level 1. Since the representations of type $\mathcal{W}(h_{(r,s,\pm)})$ are not representations of $\mathcal{W}(h_{(1,1,0)})$, the remaining nullvectors will impose additional constraints and the level 0 quotient must be zero.

(3) Finally if we consider the product of three indecomposable representations A, B, C not of type $\mathcal{W}(h_{(a,b,0)})$, we then have

$$\begin{aligned} (A \otimes B) \otimes C &= (\mathfrak{N}_A \circ \mathfrak{N}_B(\mathfrak{B}(A) \otimes \mathfrak{B}(B))) \otimes C \\ &= \mathfrak{N}_A \circ \mathfrak{N}_B \circ \mathfrak{N}_C(\mathfrak{B}(\mathfrak{N}_A \circ \mathfrak{N}_B(\mathfrak{B}(A) \otimes \mathfrak{B}(B))) \otimes \mathfrak{B}(C)) \\ &= \mathfrak{N}_A \circ \mathfrak{N}_B \circ \mathfrak{N}_C((\mathfrak{B}(A) \otimes \mathfrak{B}(B)) \otimes \mathfrak{B}(C)) \\ &= \mathfrak{N}_A \circ \mathfrak{N}_B \circ \mathfrak{N}_C(\mathfrak{B}(A) \otimes \mathfrak{B}(B) \otimes \mathfrak{B}(C)), \end{aligned} \tag{3.22}$$

$$\begin{aligned} A \otimes (B \otimes C) &= A \otimes (\mathfrak{N}_B \circ \mathfrak{N}_C(\mathfrak{B}(B) \otimes \mathfrak{B}(C))) \\ &= \mathfrak{N}_A \circ \mathfrak{N}_B \circ \mathfrak{N}_C(\mathfrak{B}(A) \otimes \mathfrak{B}(\mathfrak{N}_B \circ \mathfrak{N}_C(\mathfrak{B}(B) \otimes \mathfrak{B}(C)))) \\ &= \mathfrak{N}_A \circ \mathfrak{N}_B \circ \mathfrak{N}_C(\mathfrak{B}(A) \otimes (\mathfrak{B}(B) \otimes \mathfrak{B}(C))) \\ &= \mathfrak{N}_A \circ \mathfrak{N}_B \circ \mathfrak{N}_C(\mathfrak{B}(A) \otimes \mathfrak{B}(B) \otimes \mathfrak{B}(C)). \end{aligned} \tag{3.23}$$

Therefore, the fusion rules defined in this section are associative if the fusion rules in [18, 19] are associative².

4. The Grothendieck group

We will now study the Grothendieck group, an object closely related to open string spectra. The Grothendieck group $K_0 \equiv K_0(\text{Rep}(\mathcal{W}(p, q)))$ of representations of $\mathcal{W}(p, q)$ is, roughly speaking, the quotient set obtained by identifying two representations if they have the same character. We denote the equivalence class of a representation \mathcal{R} by $[\mathcal{R}]$. The group operation is Abelian and defined by the direct sum

$$[\mathcal{R}_1] + [\mathcal{R}_2] = [\mathcal{R}_2 \oplus \mathcal{R}_1]. \tag{4.1}$$

For example, the exact sequences (2.10) and (2.12) imply

$$[\mathcal{W}_{a,b}] = [\mathcal{W}_{a,b}^*] = [\mathcal{W}(h_{(a,b,0)})] + [\mathcal{W}(h_{(a,b,+)})]. \tag{4.2}$$

Since the characters of all indecomposable representations can be written as linear combinations of characters of irreducible representations (see appendix D), the Grothendieck group is the free Abelian group generated by the irreducible representations.

For non-logarithmic rational conformal field theories, the Grothendieck group also has a product structure turning it into a ring which is defined by

$$[\mathcal{R}_1] \cdot [\mathcal{R}_2] = [\mathcal{R}_1 \otimes \mathcal{R}_2]. \tag{4.3}$$

For the $\mathcal{W}_{p,q}$ triplet models the situation is not quite as simple; a consistent product structure can no longer be defined for the entire Grothendieck group. The counter example in [17] can be easily generalized for all (p, q) :

$$\begin{aligned} [\mathcal{W}(h_{(1,1,0)})] \cdot [\mathcal{W}_{a,b}^*] &= [\mathcal{W}(h_{(1,1,0)}) \otimes \mathcal{W}_{a,b}^*] = 0 \quad \text{versus} \\ [\mathcal{W}(h_{(1,1,0)})] \cdot [\mathcal{W}_{a,b}^*] &= [\mathcal{W}(h_{(1,1,0)})] \cdot ([\mathcal{W}(h_{(a,b,0)})] + [\mathcal{W}(h_{(a,b,+)})]) \\ &= [\mathcal{W}(h_{(1,1,0)}) \otimes \mathcal{W}(h_{(a,b,0)})] + [\mathcal{W}(h_{(1,1,0)}) \otimes \mathcal{W}(h_{(a,b,+)})] \\ &= [\mathcal{W}(h_{(a,b,0)})]. \end{aligned} \tag{4.4}$$

It was shown in [17], however, that if a representation \mathcal{M} has a dual it induces a well-defined map

$$\begin{aligned} K_0 &\rightarrow K_0 \\ [\mathcal{R}] &\mapsto [\mathcal{M} \otimes \mathcal{R}]. \end{aligned} \tag{4.5}$$

² We have checked this explicitly for $(p, q) = (2, 3)$ in [17] and this is believed to be true for all (p, q) .

We can therefore define the subgroup K_0^r of K_0 generated by $[\mathcal{R}]$ for all \mathcal{R} which have a dual representation. As in [17] we believe that these are the representations of type B together with the representations of type $\mathcal{W}_{a,b}$. K_0^r is then spanned by $\frac{1}{2}(5pq - (p + q) + 1)$ classes of representations

$$K_0^r := \text{span}_{\mathbb{Z}}([\mathcal{W}_{a,b}], [\mathcal{W}(h_{(r,q,\pm)}), [\mathcal{W}(h_{(q,s,\pm)}), [\mathcal{R}^{(2)}(h_{(a,b,+)}; h_{(p-a,b,-)}), [\mathcal{R}^{(2)}(h_{(p-a,b,-)}; h_{(a,b,+)}), [\mathcal{R}^{(2)}(h_{(a,q-b,-)}; h_{(a,b,+)})]]. \tag{4.6}$$

This is less than the total number of representations of types B and $\mathcal{W}_{a,b}$ since their characters are linearly dependent as one can see in appendix D. The basis can also be written in terms of irreducible representations, but the product then no longer corresponds to fusion. Rather one has to first perform the following substitutions before interpreting the product as fusion

$$\begin{aligned} [\mathcal{W}(h_{(a,b,0)})] &= [\mathcal{R}^{(2)}(h_{(a,b,+)}; h_{(p-a,b,-)})] - [\mathcal{R}^{(2)}(h_{(p-a,b,-)}; h_{(a,b,+)})] \\ [\mathcal{W}(h_{(a,b,+)})] &= [\mathcal{W}_{a,b}] - [\mathcal{W}(h_{(a,b,0)})] \\ &= [\mathcal{W}_{a,b}] + [\mathcal{R}^{(2)}(h_{(p-a,b,-)}; h_{(a,b,+)})] - [\mathcal{R}^{(2)}(h_{(a,b,+)}; h_{(p-a,b,-)})] \\ 2[\mathcal{W}(h_{(a,b,-)})] &= [\mathcal{R}^{(2)}(h_{(a,b,-)}; h_{(p-a,b,+)})] - 2[\mathcal{W}(h_{(p-a,b,+)})] \\ &= [\mathcal{R}^{(2)}(h_{(a,b,-)}; h_{(p-a,b,+)})] + 2[\mathcal{R}^{(2)}(h_{(p-a,b,+)}; h_{(a,b,-)})] \\ &\quad - 2[\mathcal{R}^{(2)}(h_{(a,b,-)}; h_{(p-a,b,+)})] - 2[\mathcal{W}_{p-a,b}]. \end{aligned} \tag{4.7}$$

For example, the square of $[\mathcal{W}(h_{(1,1,0)})]$ is then given by

$$\begin{aligned} [\mathcal{W}(h_{(1,1,0)})] \cdot [\mathcal{W}(h_{(1,1,0)})] &= ([\mathcal{R}^{(2)}(h_{(1,1,+)}; h_{(p-1,1,-)})] - [\mathcal{R}^{(2)}(h_{(p-1,1,-)}; h_{(1,1,+)})]) \\ &\quad \cdot ([\mathcal{R}^{(2)}(h_{(1,1,+)}; h_{(p-1,1,-)})] - [\mathcal{R}^{(2)}(h_{(p-1,1,-)}; h_{(1,1,+)})]) \\ &= [\mathcal{R}^{(2)}(h_{(1,1,+)}; h_{(p-1,1,-)})^2] + [\mathcal{R}^{(2)}(h_{(p-1,1,-)}; h_{(1,1,+)}))^2] \\ &\quad - 2[\mathcal{R}^{(2)}(h_{(1,1,+)}; h_{(p-1,1,-)}) \otimes \mathcal{R}^{(2)}(h_{(p-1,1,-)}; h_{(1,1,+)})] \\ &= 0, \end{aligned} \tag{4.8}$$

where the fusion products were evaluated using the rules in [18].

By a long but straightforward computation, one sees that

$$[\mathcal{W}(h_{(a,b,0)})] \cdot [\mathcal{W}(h_{(r,s,\mu)})] = \begin{cases} \mu \cdot \sum_{i=|r-a|+1, \text{by} 2}^{p-|a+r-p|-1} \sum_{j=|b-s|+1, \text{by} 2}^{q-|q-b-s|-1} [\mathcal{W}(h_{(i,j,0)})] \\ \text{if } (r, s) \in \{1, \dots, p-1\} \times \{1, \dots, q-1\} \\ 0 \text{ otherwise} \end{cases}, \tag{4.9}$$

i.e. the classes of representations of type $\mathcal{W}(h_{(a,b,0)})$ form the ideal

$$\mathcal{I}_0 = \bigoplus_{(a,b) \in J} \mathbb{Z}[\mathcal{W}(h_{(a,b,0)})], \tag{4.10}$$

where

$$J := \{(a, b) | 1 \leq a \leq p-1, 1 \leq b \leq q-1, qa + ps \leq pq\}. \tag{4.11}$$

It is important to remember, that a number of rank 2 and 3 representations have the same characters and therefore belong in the same equivalence class, while performing this computation.

The Grothendieck group has a direct interpretation in terms of cylinder diagrams. It is therefore interesting to consider the subgroup K_0^b generated by representations corresponding

to boundary conditions. The open string spectrum between two boundaries labelled by representations A and B is given by

$$Z(q)_{A \rightarrow B} = \text{tr}_{B \otimes A^*}(q^{L_0 - c/24}), \tag{4.12}$$

the character of $B \otimes A^*$, or more formally it only depends on the class $[B \otimes A^*]$. Restricting ourselves to the equivalence classes of representations of type B, i.e. drop $[\mathcal{W}_{a,b}]$, K_0^b is spanned by $2pq$ generators:

$$\begin{aligned} K_0^b = \text{span}_{\mathbb{Z}}([\mathcal{W}(h_{(r,q,\pm)})], [\mathcal{W}(h_{(p,s,\pm)})]) \oplus \mathcal{I}_0 \\ \oplus_{(a,b) \in J} (2\mathbb{Z}([\mathcal{W}(h_{(a,b,+)})] + [\mathcal{W}(h_{(a,q-b,-)})]) \\ \oplus 2\mathbb{Z}([\mathcal{W}(h_{(a,b,+)})] + [\mathcal{W}(h_{(p-a,b,-)})]) \\ \oplus 2\mathbb{Z}([\mathcal{W}(h_{(p-a,q-b,+)})] + [\mathcal{W}(h_{(a,q-b,-)})])). \end{aligned} \tag{4.13}$$

Since all representations of B have duals and close under fusion, K_0^b also closes under the product induced by fusion.

5. Projective representations and modular invariant partition functions

In this section we will look for projective representations. These are of particular interest to us, since it is believed [13, 24] that the bulk spectrum of these theories should be describable in terms of a quotient of

$$\bigoplus_i \mathcal{P}_i \otimes_{\mathbb{C}} \overline{\mathcal{P}}_i, \tag{5.1}$$

where the sum runs over all projective representations and the bar refers to right-movers.

Before we determine which of our $\mathcal{W}(p, q)$ representations are projective, we recall one of a number of equivalent definitions of projective representations.

Definition 5.1. A $\mathcal{W}(p, q)$ representation \mathcal{P} is projective, if given an intertwiner $f : \mathcal{P} \rightarrow \mathcal{M}'$ and a surjective intertwiner $g : \mathcal{M} \rightarrow \mathcal{M}'$, there exists an intertwiner $e : \mathcal{P} \rightarrow \mathcal{M}$ making the following diagram commute.

$$\begin{array}{ccc} & \mathcal{P} & \\ e \swarrow & & \downarrow f \\ \mathcal{M} & \xrightarrow{g} & \mathcal{M}' \end{array} \tag{5.2}$$

The irreducible representations $\mathcal{W}(h_{(p,q,\pm)})$ do not share weights with any other $\mathcal{W}(p, q)$ representations in our lists B and N; therefore, they can only have non-trivial intertwiners with themselves. This makes them promising candidates for being projective. So if we set $\mathcal{P} = \mathcal{W}(h_{(p,q,\pm)})$ in diagram (5.2) and $\mathcal{M}' \neq \mathcal{W}(h_{(p,q,\pm)})$, we have $f \equiv 0$ and diagram (5.2) commutes for $e \equiv 0$. If on the other hand, $\mathcal{P} = \mathcal{W}(h_{(p,q,\pm)}) = \mathcal{M}'$ then by Schur's lemma $f = c_f \cdot \text{id}$, $c_f \in \mathbb{C}$ and the only \mathcal{M} for which g can be surjective is $\mathcal{W}(h_{(p,q,+)})$ with $g = c_g \cdot \text{id}$, $c_g \in \mathbb{C} \setminus \{0\}$. Therefore, the diagram (5.2) commutes for $e = \frac{c_f}{c_g} \cdot \text{id}$ and the two representations $\mathcal{W}(h_{(p,q,\pm)})$ are projective.

A further property of projective representations is that their products with representations that have duals are also projective. We assume that in analogy to [17] the representations of type B, $\mathcal{W}_{a,b}$ and $\mathcal{W}_{a,b}^*$ have duals. By computing the product of all representations of type

B, $\mathcal{W}_{a,b}$ and $\mathcal{W}_{a,b}^*$ with $\mathcal{W}(h_{(p,q,\pm)})$, we find $2pq$ indecomposable representations that also ought to be projective. We denote these representations by $\mathcal{P}(h)$ where h is the weight of the irreducible representation they are a cover of:

(1) Irreducible representations

$$\mathcal{W}(h_{(p,q,\pm)}) = \mathcal{P}(h_{(p,q,\pm)});$$

(2) Rank 2 representations

$$\mathcal{R}^{(2)}(h_{(a,q,\pm)}) = \mathcal{P}(h_{(a,q,\pm)})$$

$$\mathcal{R}^{(2)}(h_{(p,b,\pm)}) = \mathcal{P}(h_{(p,b,\pm)});$$

(3) Rank 3 representations

$$\mathcal{R}^{(3)}(h_{(a,b,\pm)}) = \mathcal{P}(h_{(a,b,\pm)}).$$

This accounts for the projective covers of all representations in B and N, except for $\mathcal{W}(h_{(a,b,0)})$. In fact, none of the representations in B and N appears to be a projective cover of $\mathcal{W}(h_{(a,b,0)})$.

5.1. Modular invariant partition function

The modular transformation properties of the characters as well as a modular invariant combination of these characters are given in [20]. It was discovered in [17] that for $(p, q) = (2, 3)$, this modular invariant function can be written as

$$Z_{p,q} = \sum_{h_{(r,s,\pm)}} \dim(\text{Hom}(\mathcal{P}(h_{(r,s,\pm)}), \mathcal{P}(h_{(r,s,\pm)})))^{-2} \cdot |\chi[\mathcal{P}(h_{(r,s,\pm)})](q)|^2,$$

where $\text{Hom}(U, W)$ is the space of intertwiners from U to W . We conjecture following the arguments of [17] that the relation

$$\text{Hom}(U, V) \cong \text{Hom}(U \otimes V^*, \mathcal{W}_{1,1}^*) \tag{5.3}$$

holds for the spaces of intertwiners between representations for general (p, q) and can therefore be used to calculate $\dim \text{Hom}(\mathcal{P}(h_{(r,s,\pm)}), \mathcal{P}(h_{(r,s,\pm)}))$ using (also in analogy to [17])

$$\dim \text{Hom}(U, \mathcal{W}_{1,1}^*) = \begin{cases} 1 & U \in \{\mathcal{W}(h_{(1,1,0)}), \mathcal{W}_{1,1}, \mathcal{W}_{1,1}^*, \mathcal{W}_{p-1,q-1}, \mathcal{R}^{(3)}(h_{(1,1,+)}), \\ & \mathcal{R}^{(2)}(h_{(1,1,+)}; h_{(p-1,1,-)}), \mathcal{R}^{(2)}(h_{(1,1,+)}; h_{(1,q-1,-)})\} \\ 0 & \text{else.} \end{cases} \tag{5.4}$$

We find that

$$\dim \text{Hom}(\mathcal{P}(h_{(p,q,\pm)}), \mathcal{P}(h_{(p,q,\pm)})) = 1 \quad \dim \text{Hom}(\mathcal{P}(h_{(r,q,\pm)}), \mathcal{P}(h_{(r,q,\pm)})) = 2 \tag{5.5}$$

$$\dim \text{Hom}(\mathcal{P}(h_{(p,s,\pm)}), \mathcal{P}(h_{(p,s,\pm)})) = 2 \quad \dim \text{Hom}(\mathcal{P}(h_{(a,b,\pm)}), \mathcal{P}(h_{(a,b,\pm)})) = 4.$$

These values for the dimension of the Hom-spaces are also consistent with the conjectured embedding structures in [18]. We thus have

$$\begin{aligned} Z_{p,q} &= |\chi[\mathcal{P}(h_{(p,q,+)})](q)|^2 + |\chi[\mathcal{P}(h_{(p,q,-)})](q)|^2 \\ &+ \frac{1}{4} \left(\sum_{r=1}^p |\chi[\mathcal{P}(h_{(r,q,+)})](q)|^2 + |\chi[\mathcal{P}(h_{(r,q,-)})](q)|^2 \right) \\ &+ \frac{1}{4} \left(\sum_{s=1}^q |\chi[\mathcal{P}(h_{(p,s,+)})](q)|^2 + |\chi[\mathcal{P}(h_{(p,s,-)})](q)|^2 \right) \\ &+ \frac{1}{16} \left(\sum_{\substack{1 \leq a \leq p \\ 1 \leq b \leq q}} |\chi[\mathcal{P}(h_{(a,b,+)})](q)|^2 + |\chi[\mathcal{P}(h_{(a,b,-)})](q)|^2 \right). \end{aligned} \tag{5.6}$$

We have convinced ourselves of the modular invariance of the formula above, by extensive numerical checks, but unfortunately the general expression seems to be too unwieldy for computer algebra systems to handle and we have not yet found a way to verify it for all coprime pairs (p, q) .

6. Conclusions

In this paper we have studied the $\mathcal{W}_{p,q}$ triplet models. The structure we have found is very analogous to the results already obtained for the $\mathcal{W}_{2,3}$ models in [17]. The representations appearing in [17] were generalized for arbitrary (p, q) and we showed that the fusion rules can easily be extended to these new representations by conjecturing very plausible fusion rules for $\mathcal{W}_{1,1}^*$, $\mathcal{W}(h_{(1,1,+)})$ and $\mathcal{W}(h_{(1,1,-)})$ as well as using associativity and commutativity of the fusion product. Subsequently, the Grothendieck group K_0 was constructed together with subgroups K_0^r and K_0^b on which consistent fusion-induced products can be defined. As a final exercise, the projective representations were identified and used to suggest the structure of a modular invariant bulk theory.

This entire representations theoretic analysis suggests that a consistent boundary theory can be defined from which one can then construct a bulk theory in analogy to the Cardy case [25–29] or as was done for the $\mathcal{W}_{1,p}$ models in [13]. It would be very interesting to construct the bulk theory for example for $(p, q) = (2, 3)$. If this succeeds, this should probably directly generalize to (p, q) as the analysis of the fusion rules in this paper did.

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Appendix A. Notation

Unless stated otherwise we will always assume that

$$\begin{aligned} \alpha \in \{0, \dots, p\}, & \quad r \in \{1, \dots, p\}, & \quad a \in \{1, \dots, p-1\}, \\ \beta \in \{0, \dots, q\}, & \quad s \in \{1, \dots, q\}, & \quad b \in \{1, \dots, q-1\}. \end{aligned}$$

We will be using the notation of [20] to compactly label the weights

$$\begin{aligned} h_{(a,b,0)} &:= h_{a,b} = h_{p-a,q-b} \\ h_{(r,s,+)} &:= h_{2p-r,s} = h_{r,2q-s} = h_{r,s} + (p-r)(q-s) \\ h_{(r,s,-)} &:= h_{3p-r,s} = h_{r,3q-s} = h_{r,s} + (p-r)(q-s) + \frac{5}{4}pq - \frac{ps+qr}{2}. \end{aligned} \tag{A.1}$$

Appendix B. Dictionary to the notation in other works

B.1. The notation in [18, 19]

The fusion rules for the image of \mathfrak{B} and representations of type $\mathcal{W}_{a,b}$ are contained in [18, 19]. Here we give a dictionary between the two notations

Our notation	Notation in [18, 19]
$\mathcal{W}_{a,b}$	(a, b)
$\mathcal{W}(h_{(r,q,-(-)^\kappa)})$	$(r, \kappa q)$
$\mathcal{W}(h_{(p,s,-(-)^\kappa)})$	$(\kappa p, s)$
$\mathcal{R}^{(2)}(h_{(p-a,b,-(-)^\kappa)}; h_{(a,b,(-)^\kappa)})$	$\mathcal{R}_{\kappa p,b}^{a,0}$
$\mathcal{R}^{(2)}(h_{p-a,q,-(-)^\kappa})$	$\mathcal{R}_{\kappa p,q}^{a,0}$
$\mathcal{R}^{(2)}(h_{(a,q-b,-(-)^\kappa)}; h_{(a,b,(-)^\kappa)})$	$\mathcal{R}_{a,\kappa q}^{0,b}$
$\mathcal{R}^{(2)}(h_{(p,q-b,-(-)^\kappa)})$	$\mathcal{R}_{p,\kappa q}^{0,b}$
$\mathcal{R}^{(3)}(h_{(p-a,q-b,+)})$	$\mathcal{R}_{p,q}^{a,b}$
$\mathcal{R}^{(3)}(h_{(p-a,q-b,-)})$	$\mathcal{R}_{p,2q}^{a,b}$

Here, $\kappa = 1, 2$. The representations of type $\mathcal{W}_{a,b}^*$, $\mathcal{W}(h_{(a,b,\pm)})$ and $\mathcal{W}(h_{(a,b,0)})$ are not considered in [18, 19].

B.2. The notation in [20]

Our notation	Notation in [20]
$\mathcal{W}_{a,b}$	$\mathcal{K}_{a,b}^+$
$\mathcal{W}(h_{(a,b,0)})$	$\mathcal{X}_{a,b}$
$\mathcal{W}(h_{(a,b,+)})$	$\mathcal{X}_{a,b}^+$
$\mathcal{W}(h_{(a,b,-)})$	$\mathcal{K}_{a,b}^- = \mathcal{X}_{a,b}^-$
$\mathcal{W}(h_{(r,q,\pm)})$	$\mathcal{K}_{r,q}^\pm = \mathcal{X}_{r,q}^\pm$
$\mathcal{W}(h_{(p,s,\pm)})$	$\mathcal{K}_{p,s}^\pm = \mathcal{X}_{p,s}^\pm$

The representations of type $\mathcal{W}_{a,b}^*$ as well as the rank 2 and 3 representations are not considered in [20].

B.3. The notation in [17]

Our notation	Notation in [17]
$\mathcal{W}_{1,1}$	\mathcal{W}
$\mathcal{W}_{1,1}^*$	\mathcal{W}^*
$\mathcal{W}_{1,2}$	\mathcal{Q}
$\mathcal{W}_{1,2}^*$	\mathcal{Q}^*
$\mathcal{R}^{(2)}(h_{(a,b,+)}; h_{(p-a,b,-)})$	$\mathcal{R}^{(2)}(h_{(a,b,0)}, h_{(a,b,+)}h_{(p-a,b,-)})$
$\mathcal{R}^{(2)}(h_{(a,b,+)}; h_{(a,q-b,-)})$	$\mathcal{R}^{(2)}(h_{(a,b,0)}, h_{(a,b,+)}h_{(a,q-b,-)})$
$\mathcal{R}^{(2)}(h_{(a,q-b,-)}; h_{(a,b,+)})$	$\mathcal{R}^{(2)}(h_{(a,b,+)}, h_{(a,q-b,-)})$
$\mathcal{R}^{(2)}(h_{(p-a,b,-)}; h_{(a,b,+)})$	$\mathcal{R}^{(2)}(h_{(a,b,+)}, h_{(p-a,b,-)})$
$\mathcal{R}^{(2)}(h_{(a,q,+)})$	$\mathcal{R}^{(2)}(h_{(a,q,+)}, h_{(a,q,+)})$
$\mathcal{R}^{(2)}(h_{(p-a,b,-)})$	$\mathcal{R}^{(2)}(h_{(a,q,+)}, h_{(p-a,q,-)})$
$\mathcal{R}^{(2)}(h_{(p,b,+)})$	$\mathcal{R}^{(2)}(h_{(p,b,+)}, h_{(p,b,+)})$
$\mathcal{R}^{(2)}(h_{(p,q-b,-)})$	$\mathcal{R}^{(2)}(h_{(p,b,+)}, h_{(p,q-b,-)})$
$\mathcal{R}^{(3)}(h_{(a,b,+)})$	$\mathcal{R}^{(3)}(h_{(a,b,0)}, h_{(a,b,0)}, h_{(a,b,+)}, h_{(a,b,+)})$
$\mathcal{R}^{(3)}(h_{(a,q-b,-)})$	$\mathcal{R}^{(3)}(h_{(a,b,0)}, h_{(p-a,q-b,+)}, h_{(a,b,+)}, h_{(a,q-b,-)})$

Appendix C. Fusion for representations of type $\mathcal{W}_{a,b}$

In order to have some fusion rules at hand, we include the rules for products of representations of type $\mathcal{W}_{a,b}$ in our notation; please refer to [18, 19] for the remaining rules:

$$\begin{aligned}
 \mathcal{W}_{a,b} \otimes \mathcal{W}_{a',b'} = & \bigoplus_{\substack{i=|a-a'|+1 \\ \text{by 2}}}^{p-|p-a-a'|-1} \bigoplus_{\substack{j=|b-b'|+1 \\ \text{by 2}}}^{q-|q-b-b'|-1} \mathcal{W}_{i,j} \\
 & \oplus \bigoplus_{\substack{\alpha=a+a'-p-1 \text{ mod } 2 \\ \text{by 2}}}^{a+a'-p-1} \bigoplus_{\substack{j=|b-b'|+1 \\ \text{by 2}}}^{q-|q-b-b'|-1} \mathcal{R}^{(2)}(h_{(p-\alpha,j,+)}; h_{(\alpha,j,-)}) \\
 & \oplus \bigoplus_{\substack{\beta=b+b'-q-1 \text{ mod } 2 \\ \text{by 2}}}^{b+b'-q-1} \bigoplus_{\substack{i=|a-a'|+1 \\ \text{by 2}}}^{p-|p-a-a'|-1} \mathcal{R}^{(2)}(h_{(i,q-\beta,+)}; h_{(i,\beta,-)}) \\
 & \oplus \bigoplus_{\substack{\alpha=a+a'-p-1 \text{ mod } 2 \\ \text{by 2}}}^{a+a'-p-1} \bigoplus_{\substack{\beta=b+b'-q-1 \text{ mod } 2 \\ \text{by 2}}}^{b+b'-q-1} \mathcal{R}^{(3)}(h_{(p-\alpha,q-\beta,+)}). \tag{C.1}
 \end{aligned}$$

Here, we used the shorthand

$$\begin{aligned}
 \mathcal{R}^{(2)}(h_{(p,j,+)}; h_{(0,j,-)}) &= \mathcal{W}(h_{(p,j,+)}), & \mathcal{R}^{(2)}(h_{(i,q,+)}; h_{(i,0,-)}) &= \mathcal{W}(h_{(i,q,+)}), \\
 \mathcal{R}^{(3)}(h_{(p,q-b)}) &= \mathcal{R}^{(2)}(h_{(p,q-b)}), & \mathcal{R}^{(3)}(h_{(p-a,q)}) &= \mathcal{R}^{(2)}(h_{(p-a,q)}), \\
 \mathcal{R}^{(3)}(h_{(p,q)}) &= \mathcal{W}(h_{(p,q)}). \tag{C.2}
 \end{aligned}$$

Appendix D. Characters

The characters of the reducible but indecomposable representations can be expanded in terms of $\chi_{(a,b,\mu)}$, the characters of the irreducible representations $\mathcal{W}(h_{(a,b,\mu)})$:

$$\begin{aligned}
 \chi[\mathcal{W}_{a,b}] &= \chi[\mathcal{W}_{a,b}^*] = \chi_{(a,b,0)} + \chi_{(a,b,+)} \\
 \chi[\mathcal{R}^{(2)}(h_{(a,b,\pm)}; h_{(p-a,b,\mp)})] &= \delta_{+,\pm} \cdot \chi_{(a,b,0)} + 2\chi_{(a,b,\pm)} + 2\chi_{(p-a,b,\mp)} \\
 \chi[\mathcal{R}^{(2)}(h_{(a,b,\pm)}; h_{(a,q-b,\mp)})] &= \delta_{+,\pm} \cdot \chi_{(a,b,0)} + 2\chi_{(a,b,\pm)} + 2\chi_{(a,q-b,\mp)} \\
 \chi[\mathcal{R}^{(2)}(h_{(r,q,\pm)})] &= 2\chi_{(r,q,\pm)} + 2\chi_{(p-r,q,\mp)} \\
 \chi[\mathcal{R}^{(2)}(h_{(p,s,\pm)})] &= 2\chi_{(p,s,\pm)} + 2\chi_{(p,q-s,\mp)} \\
 \chi[\mathcal{R}^{(3)}(h_{(a,b,+)})] &= 2\chi_{(a,b,0)} + 4\chi_{(a,b,+)} + 4\chi_{(p-a,q-b,+)} + 4\chi_{(p-a,b,-)} + 4\chi_{(a,q-b,-)} \\
 \chi[\mathcal{R}^{(3)}(h_{(a,b,-)})] &= 2\chi_{(p-a,b,0)} + 4\chi_{(p-a,b,+)} + 4\chi_{(a,q-b,+)} + 4\chi_{(a,b,-)} + 4\chi_{(p-a,q-b,-)}.
 \end{aligned}$$

We therefore have the following equalities among characters and thus also among classes of the Grothendieck group K_0 :

$$\begin{aligned}
 \chi[\mathcal{R}^{(2)}(h_{(r,q,\pm)})] &= \chi[\mathcal{R}^{(2)}(h_{(r,q,\mp)})] \\
 \chi[\mathcal{R}^{(2)}(h_{(p,s,\pm)})] &= \chi[\mathcal{R}^{(2)}(h_{(p,s,\mp)})] \\
 \chi[\mathcal{R}^{(2)}(h_{(a,b,+)}; h_{(p-a,b,-)})] &= \chi_{(a,b,0)} + \chi[\mathcal{R}^{(2)}(h_{(p-a,b,-)}; h_{(a,b,+)})] \\
 \chi[\mathcal{R}^{(2)}(h_{(a,b,+)}; h_{(a,q-b,-)})] &= \chi_{(a,b,0)} + \chi[\mathcal{R}^{(2)}(h_{(a,q-b,-)}; h_{(a,b,+)})] \\
 \chi[\mathcal{R}^{(3)}(h_{(a,b,+)})] &= \chi[\mathcal{R}^{(3)}(h_{(p-a,q-b,+)})] = \chi[\mathcal{R}^{(3)}(h_{(p-a,b,-)})] \\
 &= \chi[\mathcal{R}^{(3)}(h_{(a,q-b,-)})]. \tag{D.1}
 \end{aligned}$$

Appendix E. Consistency checks

The fusion rules (3.15) predict that

$$\mathcal{W}(h_{(1,1,+)}) \otimes \mathcal{W}(h_{(1,1,+)}) = \mathfrak{N}_+ \circ \mathfrak{N}_+(\mathcal{W}_{1,1} \otimes \mathcal{W}_{1,1}) \tag{E.1}$$

$$= \mathfrak{N}_*(\mathcal{W}_{1,1}) = \mathcal{W}_{1,1}^*. \tag{E.2}$$

We have checked this for $(p, q) = (2, 3), (2, 5), (3, 4), (3, 5)$ using the NGK-algorithm [22, 23] to level 0. $\mathcal{W}(h_{(1,1,+)})$ has weight

$$h_{(1,1,+)} = h_{2p-1,1} = h_{1,2q-1} \tag{E.3}$$

and therefore nullvectors at levels $2p - 1$ and $2q - 1$. According to [31], these are given by

$$\mathcal{N}_{2p-1} = \sum_{k=1}^{2p-1} \sum_{\substack{\ell_i \geq 1 \\ \ell_1 + \dots + \ell_k = 2p-1}} \frac{((2p-1)!)^2 \left(-\frac{q}{p}\right)^{2p-1-k}}{\prod_{i=1}^{k-1} (\ell_1 + \dots + \ell_i)(2p-1-\ell_1-\dots-\ell_i)} L_{-\ell_1} \dots L_{-\ell_k} \mu, \tag{E.4}$$

$$\mathcal{N}_{2q-1} = \sum_{k=1}^{2q-1} \sum_{\substack{\ell_i \geq 1 \\ \ell_1 + \dots + \ell_k = 2q-1}} \frac{((2q-1)!)^2 \left(-\frac{p}{q}\right)^{2q-1-k}}{\prod_{i=1}^{k-1} (\ell_1 + \dots + \ell_i)(2q-1-\ell_1-\dots-\ell_i)} L_{-\ell_1} \dots L_{-\ell_k} \mu, \tag{E.5}$$

where μ is the highest weight vector of $\mathcal{W}(h_{(1,1,+)})$.

- $(p, q) = (2, 3), h_{(1,1,+)} = 2.$

The level 0 space is spanned by $\mu \otimes_{\mathbb{C}} \mu, (L_{-1}\mu) \otimes_{\mathbb{C}} \mu$ and is obtained by quotienting out $\mathcal{N}_3 \otimes_{\mathbb{C}} \mu, (L_{-1}\mathcal{N}_3) \otimes_{\mathbb{C}} \mu, (L_{-1}^2\mathcal{N}_3) \otimes_{\mathbb{C}} \mu$ and $\mathcal{N}_5 \otimes_{\mathbb{C}} \mu$. This leads to the relation

$$(L_{-1}^2\mu) \otimes_{\mathbb{C}} \mu \cong -7(L_{-1}\mu) \otimes_{\mathbb{C}} \mu - 8\mu \otimes_{\mathbb{C}} \mu. \tag{E.6}$$

The L_0 -action is then given by

$$\begin{aligned} \Delta_{1,0}(L_0)(\mu \otimes_{\mathbb{C}} \mu) &= (L_{-1}\mu) \otimes_{\mathbb{C}} \mu + 4\mu \otimes_{\mathbb{C}} \mu \\ \Delta_{1,0}(L_0)(\mu \otimes_{\mathbb{C}} \mu) &= (L_{-1}\mu)^2 \otimes_{\mathbb{C}} \mu + 5(L_{-1}\mu) \otimes_{\mathbb{C}} \mu \\ &= -2(L_{-1}\mu) \otimes_{\mathbb{C}} \mu - 8\mu \otimes_{\mathbb{C}} \mu. \end{aligned} \tag{E.7}$$

Thus, we can represent it by the matrix

$$L_0 = \begin{pmatrix} 4 & -8 \\ 1 & -2 \end{pmatrix}, \quad \text{which is conjugate to} \quad \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \tag{E.8}$$

which is consistent with $\mathcal{W}_{1,1}^*$.

- $(p, q) = (2, 5), h_{(1,1,+)} = 4.$

The level 0 space is spanned by $\mu \otimes_{\mathbb{C}} \mu, (L_{-1}\mu) \otimes_{\mathbb{C}} \mu$ and is obtained by quotienting out $\mathcal{N}_3 \otimes_{\mathbb{C}} \mu, \dots, (L_{-1}^6\mathcal{N}_3) \otimes_{\mathbb{C}} \mu$ and $\mathcal{N}_5 \otimes_{\mathbb{C}} \mu$. This leads to the relation

$$(L_{-1}^2\mu) \otimes_{\mathbb{C}} \mu \cong -13(L_{-1}\mu) \otimes_{\mathbb{C}} \mu - 32\mu \otimes_{\mathbb{C}} \mu. \tag{E.9}$$

The L_0 -action is then given by

$$\begin{aligned} \Delta_{1,0}(L_0)(\mu \otimes_{\mathbb{C}} \mu) &= (L_{-1}\mu) \otimes_{\mathbb{C}} \mu + 8\mu \otimes_{\mathbb{C}} \mu \\ \Delta_{1,0}(L_0)(\mu \otimes_{\mathbb{C}} \mu) &= (L_{-1}\mu)^2 \otimes_{\mathbb{C}} \mu + 9(L_{-1}\mu) \otimes_{\mathbb{C}} \mu \\ &= -4(L_{-1}\mu) \otimes_{\mathbb{C}} \mu - 32\mu \otimes_{\mathbb{C}} \mu. \end{aligned} \tag{E.10}$$

Thus, we can represent it by the matrix

$$L_0 = \begin{pmatrix} 8 & -32 \\ 1 & -4 \end{pmatrix}, \quad \text{which is conjugate to} \quad \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}, \tag{E.11}$$

which is consistent with $\mathcal{W}_{1,1}^*$.

- $(p, q) = (3, 4)$, $h_{(1,1,+)} = 6$.

The level 0 space is spanned by $\mu \otimes_{\mathbb{C}} \mu$, $(L_{-1}\mu) \otimes_{\mathbb{C}} \mu$ and is obtained by quotienting out $\mathcal{N}_5 \otimes_{\mathbb{C}} \mu, \dots, (L_{-1}^4 \mathcal{N}_5) \otimes_{\mathbb{C}} \mu$ and $\mathcal{N}_7 \otimes_{\mathbb{C}} \mu, \dots, (L_{-1}^2 \mathcal{N}_7) \otimes_{\mathbb{C}} \mu$. This leads to the relation

$$(L_{-1}^2 \mu) \otimes_{\mathbb{C}} \mu \cong -19(L_{-1}\mu) \otimes_{\mathbb{C}} \mu - 72\mu \otimes_{\mathbb{C}} \mu. \quad (\text{E.12})$$

The L_0 -action is then given by

$$\begin{aligned} \Delta_{1,0}(L_0)(\mu \otimes_{\mathbb{C}} \mu) &= (L_{-1}\mu) \otimes_{\mathbb{C}} \mu + 12\mu \otimes_{\mathbb{C}} \mu \\ \Delta_{1,0}(L_0)(\mu \otimes_{\mathbb{C}} \mu) &= (L_{-1}\mu)^2 \otimes_{\mathbb{C}} \mu + 13(L_{-1}\mu) \otimes_{\mathbb{C}} \mu \\ &= -6(L_{-1}\mu) \otimes_{\mathbb{C}} \mu - 72\mu \otimes_{\mathbb{C}} \mu. \end{aligned} \quad (\text{E.13})$$

Thus, we can represent it by the matrix

$$L_0 = \begin{pmatrix} 12 & -72 \\ 1 & -6 \end{pmatrix}, \quad \text{which is conjugate to } \begin{pmatrix} 0 & 0 \\ 0 & 6 \end{pmatrix}, \quad (\text{E.14})$$

which is consistent with $\mathcal{W}_{1,1}^*$.

- $(p, q) = (3, 5)$, $h_{(1,1,+)} = 8$. The level 0 space is spanned by $\mu \otimes_{\mathbb{C}} \mu$, $(L_{-1}\mu) \otimes_{\mathbb{C}} \mu$ and is obtained by quotienting out $\mathcal{N}_5 \otimes_{\mathbb{C}} \mu, \dots, (L_{-1}^6 \mathcal{N}_5) \otimes_{\mathbb{C}} \mu$ and $\mathcal{N}_9 \otimes_{\mathbb{C}} \mu, \dots, (L_{-1}^2 \mathcal{N}_9) \otimes_{\mathbb{C}} \mu$. This leads to the relation

$$(L_{-1}^2 \mu) \otimes_{\mathbb{C}} \mu \cong -25(L_{-1}\mu) \otimes_{\mathbb{C}} \mu - 128\mu \otimes_{\mathbb{C}} \mu. \quad (\text{E.15})$$

The L_0 -action is then given by

$$\begin{aligned} \Delta_{1,0}(L_0)(\mu \otimes_{\mathbb{C}} \mu) &= (L_{-1}\mu) \otimes_{\mathbb{C}} \mu + 16\mu \otimes_{\mathbb{C}} \mu \\ \Delta_{1,0}(L_0)(\mu \otimes_{\mathbb{C}} \mu) &= (L_{-1}\mu)^2 \otimes_{\mathbb{C}} \mu + 17(L_{-1}\mu) \otimes_{\mathbb{C}} \mu \\ &= -8(L_{-1}\mu) \otimes_{\mathbb{C}} \mu - 128\mu \otimes_{\mathbb{C}} \mu. \end{aligned} \quad (\text{E.16})$$

Thus, we can represent it by the matrix

$$L_0 = \begin{pmatrix} 16 & -128 \\ 1 & -8 \end{pmatrix}, \quad \text{which is conjugate to } \begin{pmatrix} 0 & 0 \\ 0 & 8 \end{pmatrix}, \quad (\text{E.17})$$

which is consistent with $\mathcal{W}_{1,1}^*$.

Note added. While this paper was being written another paper with significant overlap appeared on the arXiv [30]. In [30] the same fusion algebra is computed from a different perspective by focusing on symmetry principles. The general philosophy in this paper however, was to use associativity and commutativity to reexpress arbitrary fusion products as known products and products involving $\mathcal{W}_{1,1}^*$, $\mathcal{W}(h_{(1,1,+)})$ and $\mathcal{W}(h_{(1,1,-)})$ for which we conjectured fusion rules.

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